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# Calculation of Damage-dependent Directional Failure Indices from the Tsai-Wu Static Failure Criterion

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## Abstract

In the Special Issue of Composites Science and Technology on *Failure criteria in fibre-reinforced polymer composites* (1998), Liu and Tsai proposed an extension of the Tsai-Wu static failure criterion for progressive failure analysis (Liu KS, Tsai SW. A progressive quadratic failure criterion for a laminate. Comp Sci Tech 1998;58(7):1023-1032). They defined the strength ratio R as the scaling factor of the loading vector, and the failure index k as the inverse value of R (k = 1/R). The current paper outlines a more general approach, where the failure index k is replaced by a set of directional failure indices $\Sigma_{ij}$ (associated with the corresponding stresses $\sigma_{ij}$). Further, it will be demonstrated that the failure indices can be made damage-dependent by replacing the nominal stresses $\sigma_{ij}$ by the effective stresses $\tilde{\sigma}_{ij}$ in the Tsai-Wu static failure criterion.

**Keywords** A: Polymer-matrix composites (PMCs); B: Fracture; B: Strength; C: Failure criterion.

## 1. Introduction

Most experimental determinations of the material strength are based on uni-axial stress states. However, in real composite structures, multi-axial stress states are present and failure under these multi-axial stress states should be predicted. To that purpose, many failure criteria have been developed, amongst which the Tsai-Wu quadratic failure criterion is widely used.
The failure surface of the Tsai-Wu failure criterion can be written as [1]:

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j = 1 \qquad i, j = 1,2,...,6 \tag{1}$$

where $F_i$ and $F_{ij}$ are strength tensors of the second and fourth rank, respectively.
In its original form, the Tsai-Wu quadratic failure criterion [1] only predicts the moment of final failure, but not the failure mode. In 1998, Liu and Tsai [2] presented an extension of the Tsai-Wu failure criterion for progressive failure analysis. They defined the strength ratio R as the linear scaling factor for the loading vector. If R = 1, failure occurs. In addition, the failure index k was defined as the inverse value of R.
In the current paper, the definition of the failure index k is extended to a set of directional failure indices $\Sigma_{ij}$ which are associated with the respective stress components $\sigma_{ij}$. These non-dimensional failure indices $\Sigma_{ij}$ represent the linear scaling factors for each separate stress component $\sigma_{ij}$. Further, it is demonstrated that these failure indices can include the adverse

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effect of damage by replacing the nominal stresses $\sigma_{ij}$ by the effective stresses $\tilde{\sigma}_{ij}$ in the Tsai-Wu static failure criterion.
First, the one-dimensional formulation of the Tsai-Wu static criterion will be used to present some basic definitions. Next the multi-dimensional formulation of the Tsai-Wu quadratic failure criterion will be used to define the directional failure indices $\Sigma_{ij}$.
In this paper, only the concept and the equations are presented. The directional failure indices have been successfully used by the authors in their research on fatigue damage models for fibre-reinforced composites [3-5], but it would lead too far to explain these applications here.

## 2. One-dimensional Tsai-Wu failure criterion

In its one-dimensional formulation, the Tsai-Wu failure criterion can be written as:

$$\frac{1}{X_T \cdot |X_C|}\sigma^2 + \left(\frac{1}{X_T} - \frac{1}{|X_C|}\right)\sigma = 1 \tag{2}$$

where $X_T$ and $X_C$ are the ultimate tensile and compressive static strength, respectively.
Liu and Tsai [2] defined the strength ratio R as the positive root of the equation:

$$\left(\sigma^2 \frac{1}{X_T \cdot |X_C|}\right)R^2 + \left[\sigma\left(\frac{1}{X_T} - \frac{1}{|X_C|}\right)\right]R - 1 = 0 \tag{3}$$

It can be easily calculated that R is a linear scaling factor, equal to $X_T/\sigma$ ($\sigma \geq 0$) or $-|X_C|/\sigma$ ($\sigma < 0$). The failure index k was defined as $k = 1/R$ [2].

The definition of the failure index will now be extended so that the Tsai-Wu failure criterion can be used for a progressive failure analysis in presence of damage (e.g. fatigue, impact). Many damage models for fibre-reinforced composites are based on the Continuum Damage Mechanics theory [6-13] which can generally be defined as "*... mechanical and phenomenological models of the material degradation leading to failure and aimed at durability predictions and including mechanical weakening*" [12].
One of the basic assumptions of the Continuum Damage Mechanics theory was introduced by Lemaitre [13]. He defined the concept of strain equivalence which states that a damaged volume of material under the nominal stress $\sigma$ shows the same strain response as a comparable undamaged volume under the effective stress $\tilde{\sigma}$. Applying this principle to the elastic strain, the relation is [10]:

$$\varepsilon_e = \frac{\tilde{\sigma}}{E_0} = \frac{\sigma}{E_0(1-D)} \tag{4}$$

where $\varepsilon_e$ is the elastic strain, $\sigma$ is the nominal stress, $\tilde{\sigma}$ is the effective stress, $E_0$ is the modulus of elasticity for the undamaged material and D is a macroscopic measure of damage, lying between zero (no damage) and one (complete failure).
Now, the failure index k which is a function of the applied stress $\sigma$, is replaced by the *damage failure index* $\Sigma(\sigma, D)$. To that purpose, the nominal stress $\sigma$ is replaced by the effective stress

$\tilde{\sigma}$ (see Equation (4)) in the Tsai-Wu criterion and the corresponding *damage failure index* $\Sigma(\sigma, D)$ is calculated from:

$$\left(\frac{\sigma}{\Sigma\cdot(1-D)}\right)^2 \frac{1}{X_T\cdot|X_C|} + \frac{\sigma}{\Sigma\cdot(1-D)}\left(\frac{1}{X_T}-\frac{1}{|X_C|}\right) - 1 = 0 \tag{5}$$

It can be easily calculated that the roots of Equation (5) are: $\Sigma = \sigma/[X_T\cdot(1\text{-}D)]$ and $\Sigma = -\sigma/\left[|X_C|\cdot(1-D)\right]$. Depending on the sign of the nominal stress $\sigma$, the *damage failure index* $\Sigma(\sigma, D)$ can thus be written as:

$$\Sigma(\sigma,D) = \frac{\tilde{\sigma}}{X} = \frac{\dfrac{\sigma}{1-D}}{X} \overset{\text{Eq.(4)}}{=} \frac{E_0\cdot\varepsilon}{X} \qquad \begin{cases} X = X_T & \text{if } \sigma \geq 0 \\ X = -|X_C| & \text{if } \sigma < 0 \end{cases} \tag{6}$$

This *damage failure index* $\Sigma(\sigma, D)$ is a non-dimensional damage-dependent stress measure, because if damage increases, the stiffness decreases and the resulting strain will be higher for the same applied stress. Its value is dimensionless, and lying between zero (for $\tilde{\sigma} = \sigma = 0$) and one (for $\tilde{\sigma} = X_T$ or $\tilde{\sigma} = -|X_C|$).

The *damage failure index* $\Sigma(\sigma, D)$ has been successfully used in the authors' work on fatigue damage modelling of fibre-reinforced composites [4,5,14].

## 3. More-dimensional Tsai-Wu failure criterion

### *3.1. Definition of directional failure indices*

In order to represent the data comprehensively, the more-dimensional Tsai-Wu failure criterion is applied to the plane stress case in the following discussion. However, all definitions below can be easily applied to the complete Tsai-Wu equation (Eq. (1)).

For the plane stress case, the Tsai-Wu quadratic failure criterion can be written as:

$$F_{11}\,\sigma_1^2 + F_{22}\,\sigma_2^2 + F_{66}\,\sigma_6^2 + F_1\,\sigma_1 + F_2\,\sigma_2 + 2\,F_{12}\,\sigma_1\,\sigma_2 = 1 \tag{7}$$

In their basic papers on the Tsai-Wu static failure criterion, Tsai and Wu [1,15] calculated that $F_{12}$ may be considered zero if it falls within the range $\pm\ 0.6 \times 10^{-4}$. These conclusions were based on the strength values measured for unidirectional graphite/epoxy specimens and it is now generally accepted that the influence of the $F_{12}$-term is often negligible. Narayanaswami and Adelman [16] have studied the tensor polynomial and Hoffman strength theories for composite materials and they as well concluded that the interaction coefficient is small, and can often taken to be zero.

If $F_{12} = 0.0$, Equation (7) can be written as:

$$\frac{1}{X_T\cdot|X_C|}\sigma_{11}^2 + \frac{1}{Y_T\cdot|Y_C|}\sigma_{22}^2 + \frac{1}{S^2}\sigma_{12}^2 + \left(\frac{1}{X_T}-\frac{1}{|X_C|}\right)\sigma_{11} + \left(\frac{1}{Y_T}-\frac{1}{|Y_C|}\right)\sigma_{22} = 1 \tag{8}$$

where $X_T$ and $X_C$ are the tensile and compressive longitudinal strength, $Y_T$ and $Y_C$ are the tensile and compressive transverse strength and S is the shear strength.

The corresponding failure index $\Sigma_{11}$ for the stress component $\sigma_{11}$ is then defined as the positive root of the equation:

$$\frac{1}{X_T \cdot |X_C|}\left(\frac{\sigma_{11}}{\Sigma_{11}}\right)^2 + \frac{1}{Y_T \cdot |Y_C|}\sigma_{22}^2 + \frac{1}{S^2}\sigma_{12}^2 + \left(\frac{1}{X_T} - \frac{1}{|X_C|}\right)\frac{\sigma_{11}}{\Sigma_{11}} + \left(\frac{1}{Y_T} - \frac{1}{|Y_C|}\right)\sigma_{22} = 1 \qquad (9)$$

The failure index $\Sigma_{22}$ for the stress component $\sigma_{22}$ is defined as the positive root of the equation:

$$\frac{1}{X_T \cdot |X_C|}\sigma_{11}^2 + \frac{1}{Y_T \cdot |Y_C|}\left(\frac{\sigma_{22}}{\Sigma_{22}}\right)^2 + \frac{1}{S^2}\sigma_{12}^2 + \left(\frac{1}{X_T} - \frac{1}{|X_C|}\right)\sigma_{11} + \left(\frac{1}{Y_T} - \frac{1}{|Y_C|}\right)\frac{\sigma_{22}}{\Sigma_{22}} = 1 \qquad (10)$$

Finally, the failure index $\Sigma_{12}$ for the stress component $\sigma_{12}$ is defined as the positive root of the equation:

$$\frac{1}{X_T \cdot |X_C|}\sigma_{11}^2 + \frac{1}{Y_T \cdot |Y_C|}\sigma_{22}^2 + \frac{1}{S^2}\left(\frac{\sigma_{12}}{\Sigma_{12}}\right)^2 + \left(\frac{1}{X_T} - \frac{1}{|X_C|}\right)\sigma_{11} + \left(\frac{1}{Y_T} - \frac{1}{|Y_C|}\right)\sigma_{22} = 1 \qquad (11)$$

It is important to mention that the concept of directional failure indices does not depend on the chosen failure criterion. Also for other failure criteria than the Tsai-Wu criterion, similar directional failure indices could be defined. Further, the definitions can be applied to a wide range of fibre-reinforced composites, but to illustrate the meaning of the equations, a simple numerical example is given for the plain woven glass/epoxy lamina that was used in the authors' research [3]. The measured static strength properties of this material are listed in Table 1.

**Table 1 Measured in-plane static strengths of the glass/epoxy lamina [3].**

| | |
|---|---|
| $X_T$ [MPa] | 390.7 |
| $X_C$ [MPa] | 345.1 |
| $Y_T$ [MPa] | 390.7 |
| $Y_C$ [MPa] | 345.1 |
| S [MPa] | 100.6 |

Figure 1 illustrates the definition of the failure indices $\Sigma_{ij}$ for a bi-axial tension stress state ($\sigma_{11}$, $\sigma_{22}$) = (250 MPa, 150 MPa). The failure indices $\Sigma_{11}$ and $\Sigma_{22}$, calculated from Equations (9) and (10) respectively, equal 0.678 [-] and 0.479 [-].

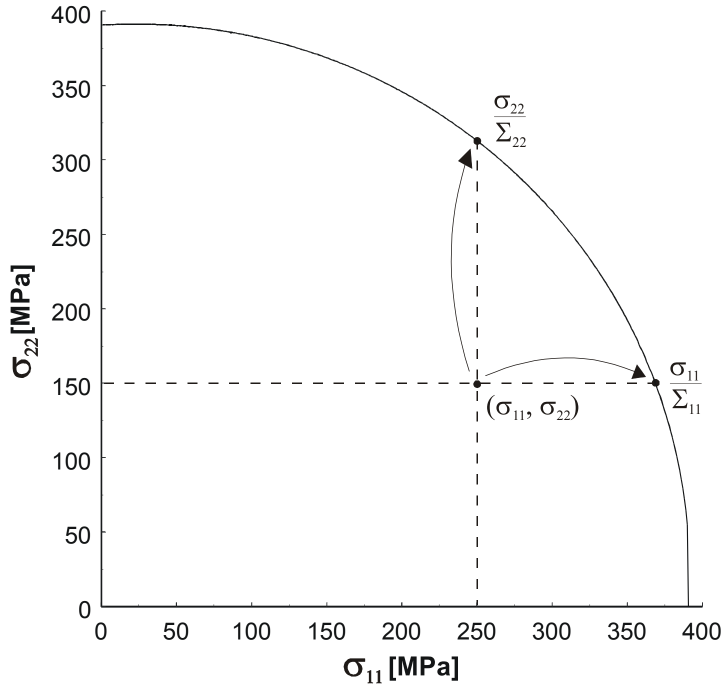


**Figure 1 Schematical representation of the definition of failure indices $\Sigma_{ij}$.**

These values for the directional failure indices $\Sigma_{11}$ and $\Sigma_{22}$ reflect the adverse effect of multi-axial loading. Indeed, if the values for $\Sigma_{11}$ and $\Sigma_{22}$ would be simply calculated as the ratio of the corresponding applied stress to the corresponding static strength, they would be 0.640 [-] (= $\sigma_{11}/X_T$) and 0.384 [-] (= $\sigma_{22}/Y_T$), respectively, being clearly lower than the values 0.678 [-] and 0.479 [-] which account for multi-axial loading.

### *3.2. Effect of the interaction term $F_{12}$*

It is worthwhile to mention that, although the interaction term $F_{12}$ has been set to zero in the discussion above, this assumption must be investigated carefully.
In the original formulation of the Tsai-Wu static failure criterion, the interaction term $2F_{12}\sigma_{11}\sigma_{22}$ is also included in the criterion. This interaction term cannot be found directly from the five strength parameters $X_T$, $X_C$, $Y_T$, $Y_C$ and S of the individual lamina, and has to be determined by bi-axial stress tests. An empirical estimation of the $F_{12}$ factor has been given by Tsai and Hahn [17]:

$$F_{12} = \frac{-\sqrt{F_{11}F_{22}}}{2} = \frac{-1}{2\sqrt{X_T \cdot |X_C| \cdot Y_T \cdot |Y_C|}} \tag{12}$$

For the glass/epoxy lamina mentioned above [3], this results in a value of $F_{12}$ being $-3.708\times10^{-6}$. Although this value is very small, the $F_{12}$-term does significantly affect the shape of the Tsai-Wu failure surface for the plain woven glass/epoxy lamina, which is far less anisotropic than the unidirectional graphite/epoxy specimens considered by Tsai and Wu [1,15].
Figure 2 shows the Tsai-Wu failure envelope for a bi-axial stress state ($\sigma_{11}$, $\sigma_{22}$) with and without the interaction term $2F_{12}\sigma_{11}\sigma_{22}$.

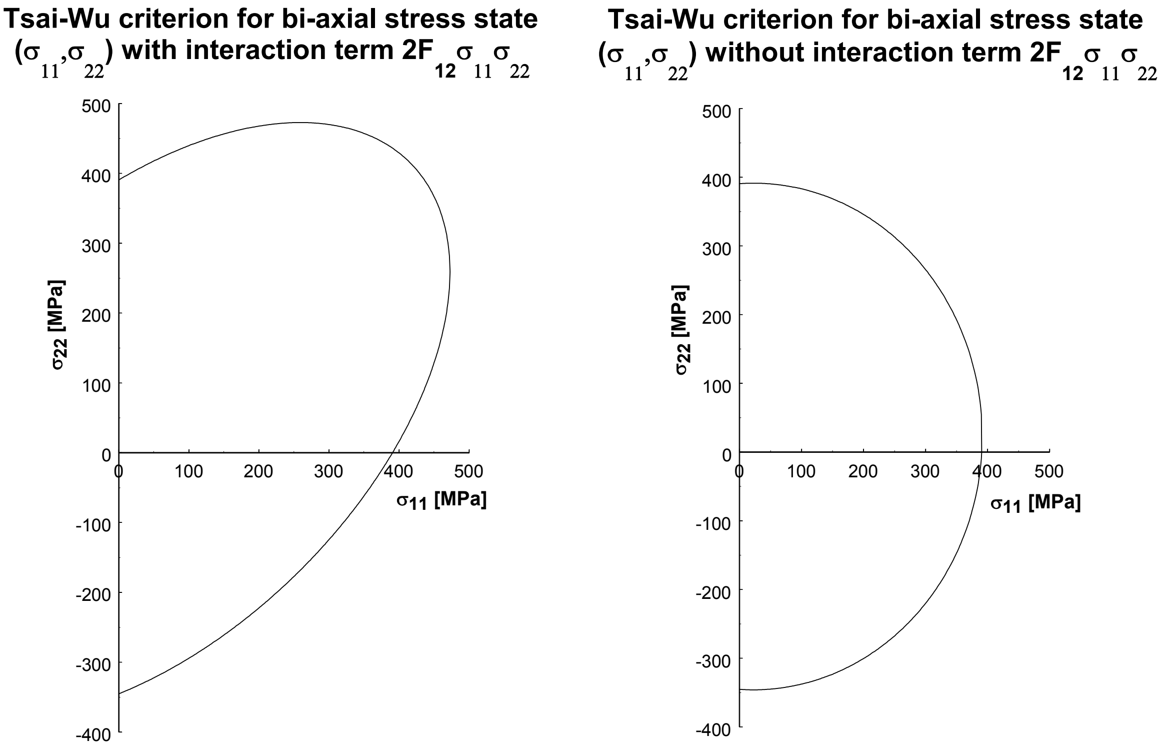


**Figure 2** **Tsai-Wu criterion for bi-axial loading of a glass fabric/epoxy lamina with and without interaction term $2F_{12}\sigma_{11}\sigma_{22}$.**

The large difference between these failure surfaces is only due to the value of the interaction term $F_{12}$. The interaction term $F_{12}$ can be determined for example from a bi-axial tension test with $\sigma_{11} = \sigma_{22} = \sigma$ and all other stresses being zero. When the bi-axial failure stress $\sigma$ has been experimentally determined, the interaction term $F_{12}$ can be calculated as [18]:

$$F_{12} = \frac{1}{2\sigma^2}\left[1-\left(\frac{1}{X_T}-\frac{1}{|X_C|}+\frac{1}{Y_T}-\frac{1}{|Y_C|}\right)\cdot\sigma-\left(\frac{1}{X_T\cdot|X_C|}+\frac{1}{Y_T\cdot|Y_C|}\right)\cdot\sigma^2\right] \tag{13}$$

The value of $F_{12}$ that would be obtained for a bi-axial failure stress ranging from 50 MPa to 350 MPa is plotted in Figure 3.

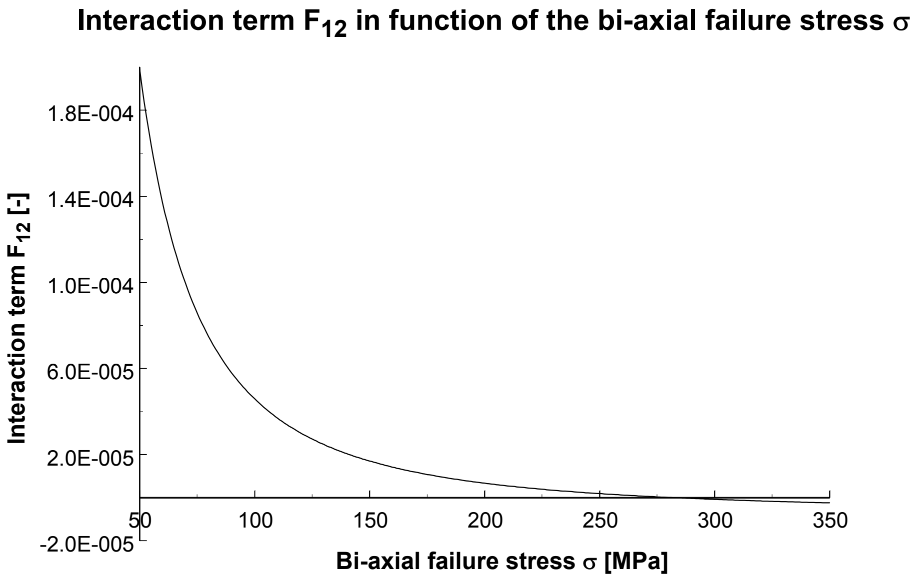


**Figure 3 Dependence of the $F_{12}$-value on the bi-axial failure stress $\sigma = \sigma_{11} = \sigma_{22}$.**

The interaction term $F_{12}$ becomes negative for a bi-axial failure stress $\sigma \approx 285$ MPa, and as can be seen from Figure 3, the variation of the $F_{12}$ value is rather large.
Although it turned out that the zero-value for $F_{12}$ was eventually a good choice [3], it must be kept in mind that the value of the $F_{12}$ factor must be investigated for this type of materials, and eventually must be included in the Equations (8)-(11).

### *3.3. Failure conditions*

If the stress vector ($\sigma_{11}$, $\sigma_{22}$, $\sigma_{12}$) approaches the failure surface, all failure indices $\Sigma_{ij}$ are approaching their failure value 1.0, although one stress component can be (mainly) responsible for that failure. This can be easily seen from Figure 1: if the stress state ($\sigma_{11}$, $\sigma_{22}$) is situated very closely to the failure surface, the reserves to failure along the two orthogonal directions $\vec{e}_{11}$ and $\vec{e}_{22}$ are both very small, because the two orthogonal distances to the failure surface are very small. On the other hand, it is possible that one of the material directions is still able to carry loads. This is illustrated by the numerical data in Table 2. If $\sigma_{11}$ in Figure 1 is increased up to 368.8 MPa, while $\sigma_{22}$ is kept constant at 150 MPa, failure occurs and both failure indices are equal to 1.0. So the failure indices do not give information about which stress component is responsible for the failure.

**Table 2 In-plane stresses and failure indices for a given stress state ($\sigma_{11}$, $\sigma_{22}$).**

| | | | | | |
|---|---|---|---|---|---|
| $\sigma_{11}$ [MPa] | 368.8 | $\Sigma_{11}$ [-] | 1.0 | $\sigma_{11}/X_T$ [-] | 0.944 |
| $\sigma_{22}$ [MPa] | 150.0 | $\Sigma_{22}$ [-] | 1.0 | $\sigma_{22}/Y_T$ [-] | 0.384 |

To assess the *relative importance of the separate stress components* $\sigma_{ij}$ in the failure event, it is better to correlate the failure indices $\Sigma_{11}$ and $\Sigma_{22}$ with their one-dimensional equivalent, because the reserve to failure for the stress $\sigma_{22}$ ($\sigma_{22}/Y_T = 0.384$) is much larger than for the stress $\sigma_{11}$ ($\sigma_{11}/X_T = 0.944$) in this example. Finally, the failure indices $\Sigma_{11}$ and $\Sigma_{22}$ must reduce to their one-dimensional equivalent if the other stress component is zero.
A definition which satisfies these requirements, is the following:

$$\begin{aligned} \Sigma_{11} &= \frac{\Sigma_{11}^{2D}}{1+(\Sigma_{11}^{2D}-\Sigma_{11}^{1D})} \\ \Sigma_{22} &= \frac{\Sigma_{22}^{2D}}{1+(\Sigma_{22}^{2D}-\Sigma_{22}^{1D})} \end{aligned} \qquad (14)$$

The failure indices $\Sigma_{11}^{2D}$ and $\Sigma_{22}^{2D}$ are calculated from the respective equations (9) and (10), while the one-dimensional failure indices $\Sigma_{11}^{1D}$ and $\Sigma_{22}^{1D}$ are defined as the ratio of the stress $\sigma$ to the respective static strength.
So, the two-dimensional failure indices $\Sigma_{ii}^{2D}$ $(i=1,2)$ take into account the adverse effect of multi-axial loading, while the correlation with the one-dimensional ratio $\Sigma_{ii}^{1D}$ $(i=1,2)$ to their respective static strengths indicates the relative probability of failure along the considered direction $\vec{e}_{11}$ or $\vec{e}_{22}$. It is important to note that the newly defined failure indices reduce to their one-dimensional equivalent if a one-dimensional stress is applied, so the relation between one-dimensional and multi-dimensional failure indices remains consistent in use.
Figure 1 can be transformed to explain the extended definition, as shown in Figure 4.

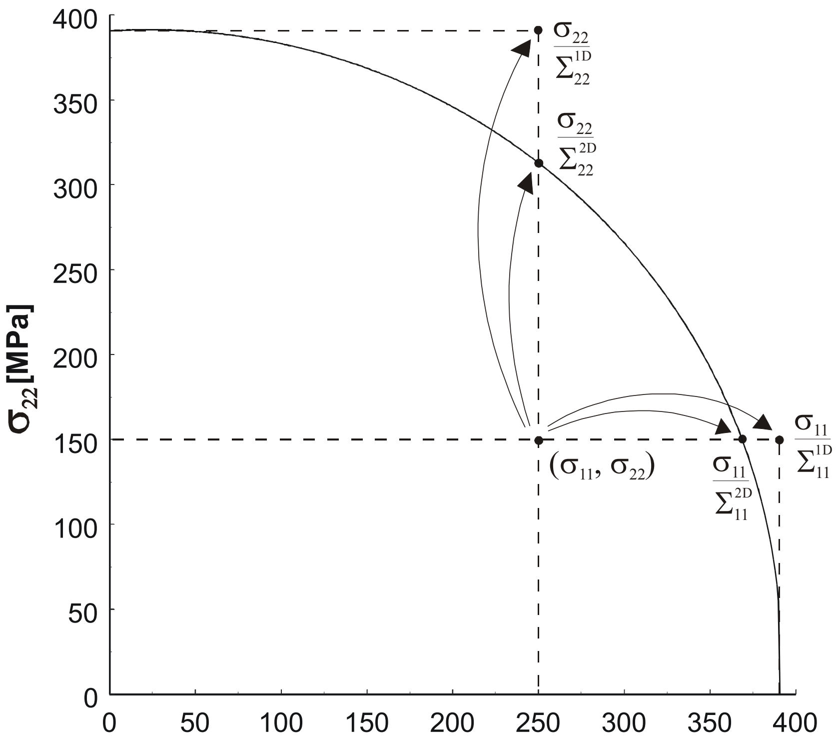

**Figure 4 Schematical representation of the definition of failure indices $\Sigma_{ij}$.**

For the numerical example in Table 2, the new failure indices $\Sigma_{11}$ and $\Sigma_{22}$ can be calculated as:

$$
\begin{aligned}
\Sigma_{11} &= \frac{\Sigma_{11}^{2D}}{1+(\Sigma_{11}^{2D}-\Sigma_{11}^{1D})} = \frac{1.0}{1.0+(1.0-0.944)} = 0.947 \\
\Sigma_{22} &= \frac{\Sigma_{22}^{2D}}{1+(\Sigma_{22}^{2D}-\Sigma_{22}^{1D})} = \frac{1.0}{1.0+(1.0-0.384)} = 0.619
\end{aligned} \tag{15}
$$

These values indicate that $\Sigma_{11}$ will be mainly responsible for failure in this particular example. It is commonly argued that the Tsai-Wu failure criterion cannot discriminate between different failure modes, but it appears that this modified definition of the failure indices remedies this problem in a certain sense: it is at least possible to predict which stress component is mainly responsible for failure.

### *3.4. Damage-dependent failure indices*

Analogous to the one-dimensional formulation, the nominal stresses $\sigma_{ij}$ can be replaced by the effective stresses $\tilde{\sigma}_{ij}$ in the Equations (9)-(11) in order to account for fatigue damage or impact damage in the calculation of the directional failure indices $\Sigma_{ij}$. Figure 4 can be transformed to explain the extended definition, as shown in Figure 5, if for instance the effective stresses $\tilde{\sigma}_{ij}$ are defined as $\sigma_{ij}/(1-D_{ij})$, with $D_{ij}$ being the damage caused by the respective stress components $\sigma_{ij}$.

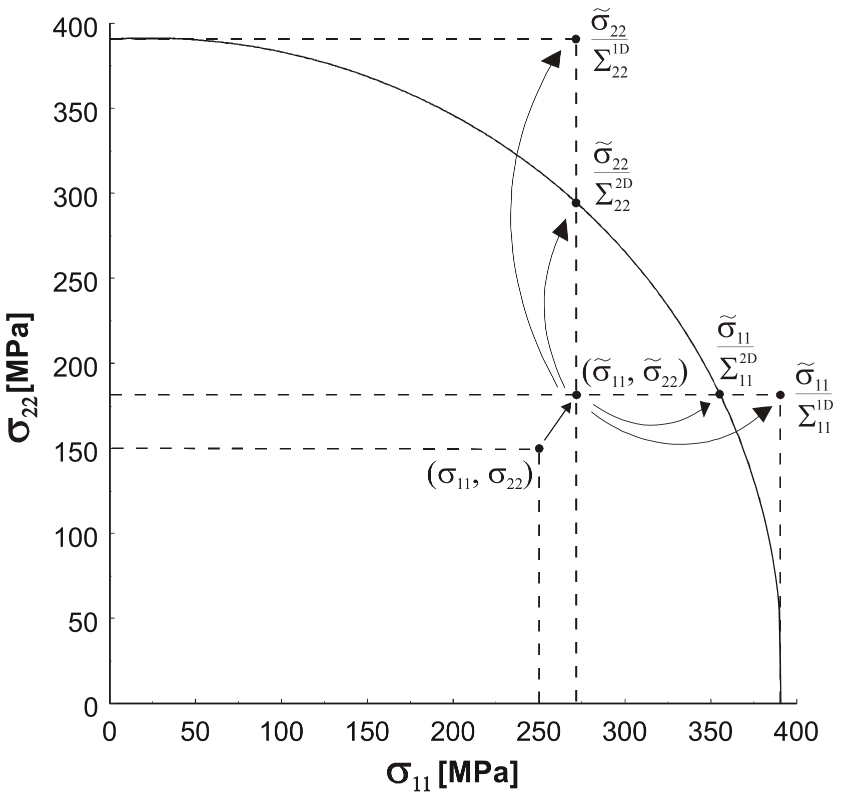

**Figure 5** **New definition of the damage-dependent failure indices.**

## 4. Case study

This paragraph briefly reviews part of the first author's doctoral research [3] to show how the damage-dependent failure indices can be used in practice.
The study was concerned with the fatigue damage modelling of fibre-reinforced polymers. In this context it is important to mention that the material used was a plain woven glass fabric/epoxy composite. This material was loaded in a *displacement-controlled* bending fatigue test. Due to the introduction of damage, the force necessary to bend the specimen decreased during fatigue life. Further, the out-of-plane displacement profile in bending changed during the test, and for some stacking sequences, a considerable permanent deformation remained after unloading.
It was the objective to predict these phenomena by simulating the full loading history of the specimen, from the first loading cycle up to final failure. To that purpose, a fatigue damage model was developed and integrated into a commercial finite element code. This model was applied to each Gauss-point of the finite element mesh and predicted the damage growth due to the applied fatigue loading in that particular Gauss-point.
The main problem to deal with was the choice of the damage variables. Basically, two options were possible:

- the damage variables are defined on the micromechanical level. Distinction is made between different damage types: matrix cracks, fibre/matrix debonding, meta-delaminations (at cross-over points of the fabric), fibre fracture,
- the damage variables are defined at the mesoscopic level. The plain woven glass fabric/epoxy lamina is treated as a homogenized orthotropic layer with directional damage.

For the first option, individual stresses in the fibre and matrix material should be calculated, as the damage growth would depend on their value. As a consequence, the finite element mesh should model the individual fibre bundles and matrix areas in the plain woven glass fabric/epoxy lamina. This requires a huge number of finite elements, and as the objective was to simulate the full loading history, the computation time would be too high.

Therefore the second option was preferred. And then, the damage-dependent failure indices can be used. Indeed, in the developed fatigue damage model, three damage variables were assigned to each Gauss-point: $D_{11}$, $D_{22}$ and $D_{12}$. These damage variables account for the intra-layer damage (matrix cracks, fibre/matrix debonding, fibre fracture) caused by the respective orthotropic stresses $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{12}$. To predict damage growth, additional growth rate equations should be established:

$$\begin{aligned} \frac{dD_{11}}{dN} &= f_1(\sigma_{ij}, D_{ij}) \\ \frac{dD_{22}}{dN} &= f_2(\sigma_{ij}, D_{ij}) \qquad (i, j = 1, 2) \\ \frac{dD_{12}}{dN} &= f_3(\sigma_{ij}, D_{ij}) \end{aligned} \tag{16}$$

where N represents the number of fatigue loading cycles.
To account for the adverse effect of a multi-axial stress state, the damage-dependent failure indices from the Tsai-Wu failure criterion have been introduced in the damage growth rate equations as follows [3]:

$$\begin{aligned} \frac{dD_{11}}{dN} &= f_1(\Sigma_{11}, D_{ij}) \\ \frac{dD_{22}}{dN} &= f_2(\Sigma_{22}, D_{ij}) \qquad (i, j = 1, 2) \\ \frac{dD_{12}}{dN} &= f_3(\Sigma_{12}, D_{ij}) \end{aligned} \tag{17}$$

whereby $\Sigma_{11}$, $\Sigma_{22}$ and $\Sigma_{22}$ are calculated from the respective equations (9)-(11), with $\sigma_{ij}$ being replaced by $\sigma_{ij}/(1-D_{ij})$ and $D_{ij}$ being the damage caused by the respective stress components $\sigma_{ij}$.
This formulation has several advantages:

- the failure indices account for the adverse effect of multi-axial fatigue loading,
- the actual value of the damage variables $D_{11}$, $D_{22}$ and $D_{12}$ in each Gauss-point is taken into account by replacing $\sigma_{ij}$ by $\sigma_{ij}/(1-D_{ij})$ in the equations (9)-(11),
- the failure indices reduce to their one-dimensional equivalent if only one stress component is active. As such, the multi-axial fatigue damage model converts to its one-dimensional equivalent in case of uni-axial fatigue loading,

This fatigue damage model has been validated for a wide range of bending fatigue tests and fatigue lives ranging from 20,000 cycles to more than 1,000,000 cycles [3]. The use of the damage-dependent failure indices provided a reliable means to perform finite element simulations with a reasonable computation time and a very good accuracy.

## 5. Conclusions

The idea of Liu and Tsai [2] to calculate a strength ratio R and a failure index k from the Tsai-Wu quadratic failure criterion, has been extended to directional failure indices $\Sigma_{ij}$. Each of these failure indices $\Sigma_{ij}$ represents the directional failure index associated with $\sigma_{ij}$. By using the effective stresses $\tilde{\sigma}_{ij}$ instead of the nominal stresses $\sigma_{ij}$, the notion of damage can be introduced into the definition of the failure indices $\Sigma_{ij}$.
The authors are well aware that the Tsai-Wu static failure criterion was not developed for this purpose and that it basically does not provide any information about the actual damage and failure mechanisms. In the particular case of the fatigue damage research, the definition of the failure indices was in fact an “engineering solution” to a complicated problem, but it appeared to be very reliable in predicting the fatigue damage behaviour of plain woven glass fabric/epoxy composites.
Finally, the authors would like to stress the fact that this concept of directional failure indices does not apply uniquely to the Tsai-Wu quadratic failure criterion. Other failure criteria could be used to define these directional failure indices $\Sigma_{ij}$ and it is perfectly possible to choose failure criteria that do provide information about the underlying damage mechanisms.

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