

Nonlinear correction to the bending stiffness of a damaged composite beam

W. Van Paepegem^{a*}, R. Dechaene^a and J. Degrieck^a

^a Professor, Dept. of Mechanical Construction and Production, Sint-Pietersnieuwstraat 41, 9000 Gent, Belgium

Abstract

When studying damage in composite materials, the classic beam theory is sometimes used in a modified manner to calculate the bending response of a damaged composite laminate. The longitudinal stiffness E_0 is then replaced by a field variable $E(x,y) = E_0 \times [1-D(x,y)]$. The damage distribution $D(x,y)$ affects the calculation of stresses and strains, and requires a modified calculation of the neutral fibre $y_0(x)$ and the bending stiffness $EI(x)$. It seems obvious to postulate that the bending stiffness $EI(x)$ is also multiplied with $[1-D(x,y)]$, as was proposed by, amongst others, Sidoroff and Subagio (Sidoroff, F. and Subagio, B. (1987). Fatigue damage modelling of composite materials from bending tests. In : Proceedings of ICCM-VI & ECCM-II, pp. 4.32-4.39).

It is shown here that this formulation is not correct and that a nonlinear correction term to the bending stiffness degradation is necessary. This term is especially important when the damage $D(x,y)$ is reaching higher values. The results of a semi-analytical simulation for cantilever beams is shown to support the findings.

Keywords: beam theory, damage, composite, bending, residual stiffness

1 Introduction

Today, a lot of research is dedicated to the understanding of damage in fibre-reinforced composites. Simple and more advanced simulation tools are used to model the mechanical response of damaged composites.

One of the straightforward calculation tools is the classic beam theory. In 1987, Sidoroff and Subagio [1] proposed a local damage model to predict the damage growth in composites subject to bending fatigue tests. This local damage model was combined with an adapted formulation of the classic beam theory to calculate the stresses and strains in the damaged composite beam under fatigue loading.

In this paper it is demonstrated that the linear relation between the degradation of the bending stiffness and the damage is not correct and a nonlinear correction term should be added. The relevant equations are derived and the formulations are implemented in a semi-analytical simulation of a bending fatigue test.

2 Review of classic beam theory for damaged beams

In this paragraph, the existing approach for damage modelling with the classic beam theory is discussed. The work of Sidoroff and Subagio is taken as a guideline as they worked out the equations and applied the theory to the bending fatigue of a composite beam.

* Author to whom correspondence should be addressed (E-mail : Wim.VanPaepegem@UGent.be).

The local damage model, proposed by Sidoroff and Subagio [1], was a one-dimensional damage model based on the Continuum Damage Mechanics theory [2-8]. The well-known relation between stress and strain was:

$$\sigma = E_0 \cdot (1 - D) \cdot \varepsilon \quad (1)$$

with E_0 the virgin elastic modulus and D the scalar damage ($0 \leq D \leq 1$). This relation evolves from the concept of strain equivalence, introduced by Lemaitre [9], which states that a damaged volume of material under the nominal stress σ shows the same strain response as a comparable undamaged volume under the effective stress $\tilde{\sigma}$ ($= \sigma/(1-D)$).

The damage growth under fatigue loading was established as follows [1]:

$$\frac{dD}{dN} = \begin{cases} \frac{A \cdot (\Delta\varepsilon)^c}{(1-D)^b} & \text{in tension} \\ 0 & \text{in compression} \end{cases} \quad (2)$$

where : - D : local damage variable
 - N : number of cycles
 - $\Delta\varepsilon$: strain amplitude of the applied cyclic loading
 - A , b and c : three material constants

The relevant equations for the bending response of the damaged composite beam were derived from classic beam theory as follows: when the axial force is supposed to remain zero and when only a bending moment exists, the position of the neutral fibre $y_0(x)$ at each moment of time is calculated as:

$$y_0(x) = \frac{\int_{-\frac{h}{2}}^{+\frac{h}{2}} [1 - D(x, y)] \cdot y \, dy}{\int_{-\frac{h}{2}}^{+\frac{h}{2}} 1 - D(x, y) \, dy} \quad (3)$$

where : - x : coordinate along the beam length
 - y : thickness-coordinate, with $y = 0$ in the middle of the specimen's thickness
 - h : total thickness of the specimen

The degraded bending stiffness $EI(x)$ becomes (with 'b' the specimen's width):

$$EI(x) = b \cdot E_0 \cdot \int_{-\frac{h}{2}}^{+\frac{h}{2}} [1 - D(x, y)] \cdot y^2 \, dy \quad (4)$$

In the next paragraph it will be shown that this latter equation is not correct and should include a nonlinear correction term for the bending stiffness $EI(x)$.

3 Derivation of the nonlinear correction term for the bending stiffness EI(x)

For the derivation of the nonlinear correction term for the bending stiffness EI(x), it is again assumed that:

$$\sigma_{xx} = E_0 \cdot (1 - D) \cdot \varepsilon_{xx} \quad (5)$$

where D is a field variable D(x,y). The x-coordinate is again along the beam length, while the y-coordinate represents the beam thickness.

The following displacement field is proposed for the bending beam:

$$\begin{cases} u_y(x, y) = u(x) \\ u_x(x, y) = u^0(x) + u' \cdot y \end{cases} \quad (6)$$

where u(x) is the vertical displacement of the bending beam, u⁰(x) is the longitudinal displacement of the midplane and y is the thickness coordinate lying in the interval [-h/2, +h/2]. The width b is assumed to be unity for the moment.

The strain ε_{xx} can then be calculated as:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{du^0}{dx} + u'' \cdot y = \varepsilon^0 + u'' \cdot y \quad (7)$$

And the corresponding stress σ_{xx} equals:

$$\sigma_{xx} = E \cdot \varepsilon_{xx} = E_0 \cdot (1 - D) \cdot (\varepsilon^0 + u'' \cdot y) \quad (8)$$

The normal force N is assumed to be zero in pure bending:

$$N = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dy = E_0 \cdot \varepsilon^0 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - D) dy + E_0 \cdot u'' \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - D) \cdot y dy = 0 \quad (9)$$

From Equation (9), the strain ε_0 can be calculated as:

$$\varepsilon_0 = -u'' \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - D) \cdot y dy}{\int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - D) dy} \quad (10)$$

Then the position of the neutral fibre y_0 is determined from the equations (7) and (10):

$$\varepsilon_{xx} = u'' \left(y - \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} (1-D) \cdot y \, dy}{\int_{-\frac{h}{2}}^{\frac{h}{2}} (1-D) \, dy} \right) = u''(y - y_0) \quad (11)$$

The normal stress σ_{xx} can then be written as:

$$\sigma_{xx} = E_0 \cdot (1-D) \cdot u'' \cdot (y - y_0) \quad (12)$$

The bending moment $M(x)$ is calculated from the expression:

$$M(x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} \cdot y \, dy \quad (13)$$

Taking into account Equation (12), the bending moment $M(x)$ can be written as:

$$M(x) = E_0 \cdot u'' \cdot \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} (1-D) \cdot y^2 \, dy - y_0 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} (1-D) \cdot y \, dy \right) \quad (14)$$

The bending stiffness $EI(x)$ is then defined from the equation:

$$M(x) = EI(x) \cdot u'' \quad (15)$$

Hence:

$$EI(x) = E_0 \cdot \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} (1-D) \cdot y^2 \, dy - \frac{\left(\int_{-\frac{h}{2}}^{\frac{h}{2}} (1-D) \cdot y \, dy \right)^2}{\int_{-\frac{h}{2}}^{\frac{h}{2}} (1-D) \, dy} \right] \quad (16)$$

If Equation (16) is compared with Equation (4) proposed by Sidoroff and Subagio [1], it is clear that a nonlinear correction term appears in the expression for the bending stiffness $EI(x)$. From Equation (11) and (15), it follows directly that:

$$\varepsilon_{xx} = \frac{M(x) \cdot (y - y_0)}{EI(x)} \quad (17)$$

4 Simulations

To show the importance of the nonlinear correction term in Equation (16), a semi-analytical simulation is performed for cantilever bending fatigue tests on a composite specimen. In Figure 1, the conventions for the simulations are shown. The composite specimen is modelled as a cantilever beam, and the moving clamp (right) is represented by a rigid rod. The prescribed displacement in the displacement-controlled bending fatigue test is $u_{\max} = u(C)$.

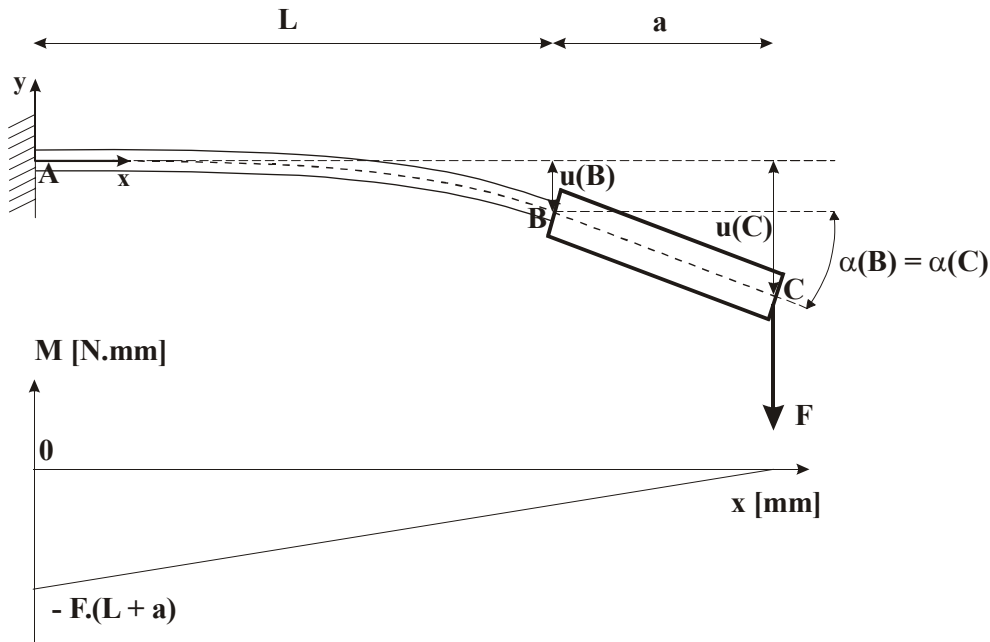


Figure 1 Bending of the cantilever composite beam following the conventions of the classic beam theory.

The force necessary to impose the prescribed displacement $u_{\max} = u(C)$, can be calculated from the following equations:

$$u'' = \frac{M(x)}{EI(x)} = -F \cdot \frac{L+a-x}{EI(x)} \quad (18)$$

$$u'(x) = -F \cdot \int_0^x \frac{L+a-x}{EI(x)} dx \quad (19)$$

$$u'(B) = -F \cdot \int_0^L \frac{L+a-x}{EI(x)} dx \quad (20)$$

$$u(B) = \int_0^L u'(x) dx = -F \cdot \int_0^L dx \int_0^x \frac{L+a-x'}{EI(x')} dx' \quad (21)$$

$$u(C) = u'(B) \cdot a + u(B) \quad (22)$$

$$u_{\max} = u(C) = -F \cdot \left(a \cdot \int_0^L \frac{L+a-x}{EI(x)} dx + \int_0^L dx \int_0^x \frac{L+a-x'}{EI(x')} dx' \right) \quad (23)$$

The equation (23) provides the relation between the force F and the prescribed displacement u_{\max} .

The local damage model is very similar to the one used by Sidoroff and Subagio and has the following form:

$$\frac{dD(x,y)}{dN} = \begin{cases} \frac{A \cdot \left(\frac{\Delta\sigma(x,y)}{\sigma_{TS}} \right)^c}{(1-D(x,y))^b} & \text{in tension} \\ 0 & \text{in compression} \end{cases} \quad (24)$$

Here, the stress amplitude $\Delta\sigma(x,y)$ in the damage law (24) corresponds with the maximum stress in each material point during the simulated fatigue loading cycle. The material parameters E_0 and σ_{TS} of the (unidirectional) composite laminate were respectively 24.57 GPa and 390.7 MPa [10]. The constants A , b and c were respectively $70.3 \cdot 10^{-3}$ [1/cycle], 0.45 [-] and 6.5 [-] [10].

Unlike the local damage model by Sidoroff and Subagio [1], this model includes an elastic limit of the stress and can be more easily related with S-N curves by use of the stress amplitude instead of the strain amplitude. Of course, if one and the same damage model is applied for both the linear and nonlinear bending stiffness degradation, it does not matter really which damage model is chosen to demonstrate the effect of the nonlinear correction term.

This constitutive model can be easily implemented in a mathematical software package such as Mathcad™. The numerical integration formulae must be chosen such that the second degree polynomials are exactly integrated. This is the case for the Simpson's rule, which is a Newton-Cotes quadrature formula. Because the increase of the damage variable D during one cycle is so small, the integrations must be exact indeed, otherwise the relative error on the calculation of the bending stiffness EI may be larger than the increase of the damage variable itself. Therefore the conventional first-order trapezium method is not suited for this purpose.

First the distribution of the bending moment along the length of the specimen is determined. Secondly the stresses and strains in each integration point can be calculated. The damage law is applied and a new cycle is evaluated. From equation (23) the necessary force to impose the displacement with amplitude u_{\max} can be calculated for each cycle. This algorithm must be solved for each calculated loading cycle, and as a consequence, the Mathcad™ worksheet must be called several times. This is only possible within the Mathconnex™ environment which allows to repetitively call the same Mathcad™ worksheet with different starting values.

5 Simulations

In the following semi-analytical simulation, the displacement-controlled bending fatigue test, shown in Figure 1, is simulated for about 300,000 loading cycles. The displacement amplitude u_{\max} was chosen to be 30.4 mm.

The simulations with linear and nonlinear bending stiffness degradation $EI(x)$ were done with the *same* material parameters E_0 and σ_{TS} and the *same* constants A , b and c for the damage law (24). The only difference is due to the nonlinear correction term $EI(x)$.

Figure 2 shows the simulated force-cycle history with linear and nonlinear bending stiffness degradation $EI(x)$. The initial force $F = 106.68$ N is the force amplitude necessary to bend the specimen into its maximum deformed state ($u = u_{\max}$) during the first loading cycle. Then, due to the damage growth, the force amplitude is decreasing during fatigue life. It is clear that the simulated degradation of the composite specimen is much larger in the case of the nonlinear correction term, although the applied fatigue damage law (24) and its constants are the same in both cases.

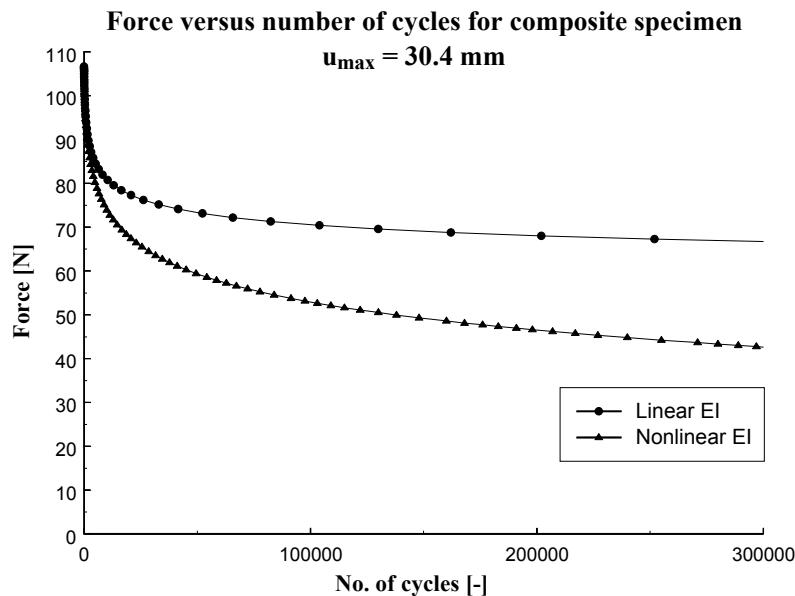


Figure 2 Simulated force-cycle history with linear and nonlinear bending stiffness degradation $EI(x)$.

Of course, the difference in force-cycle history should be reflected in a difference in damage distribution as well. This is confirmed by Figure 3 and Figure 4 which show the simulated damage distribution in the clamped cross-section ($x = 0$) with linear and nonlinear bending stiffness degradation respectively.

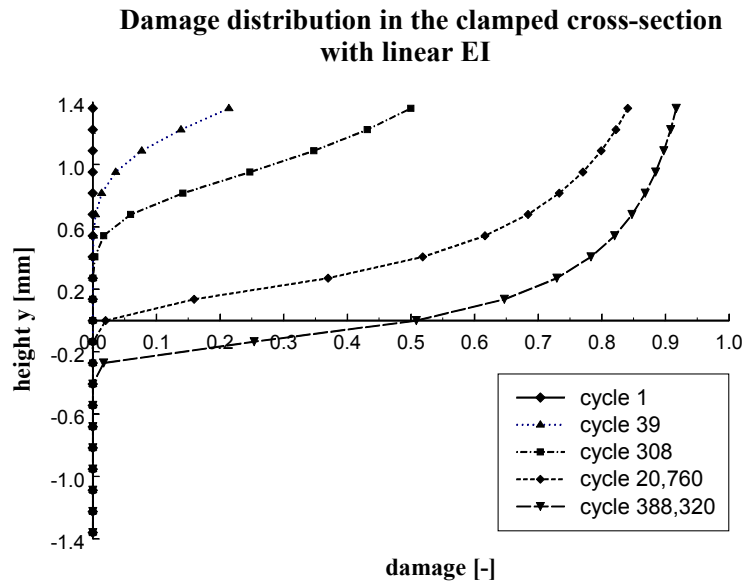


Figure 3 Simulated damage distribution in the clamped cross-section at several stages during fatigue life with linear EI.

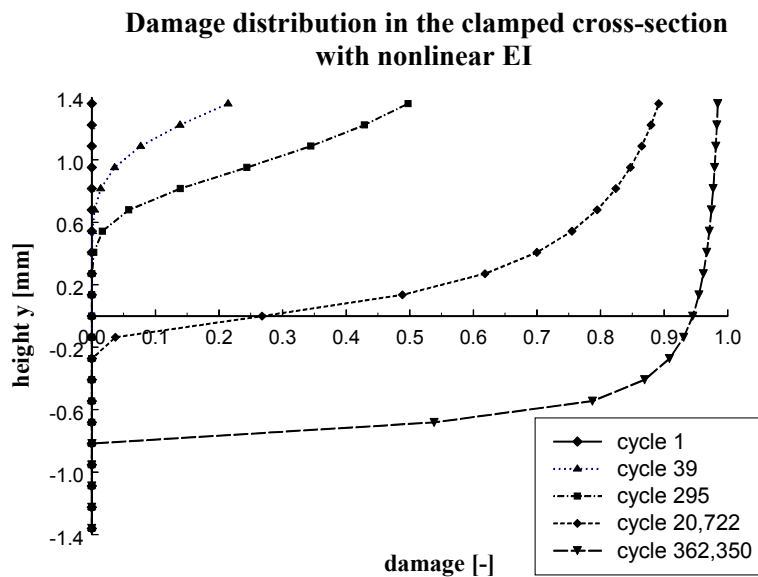


Figure 4 Simulated damage distribution in the clamped cross-section at several stages during fatigue life with nonlinear EI.

Figure 5 and Figure 6 show the corresponding simulated strain distribution in the clamped cross-section with linear and nonlinear bending stiffness degradation. For the simulation without the correction term, the strain increases at the tensile side of the specimen, but decreases at the compressive side. This is not realistic as due to the increasing damage at the tensile side, stress redistribution occurs and the strain should increase at the compressive side. This is confirmed by the simulation in Figure 6 with the nonlinear correction term.

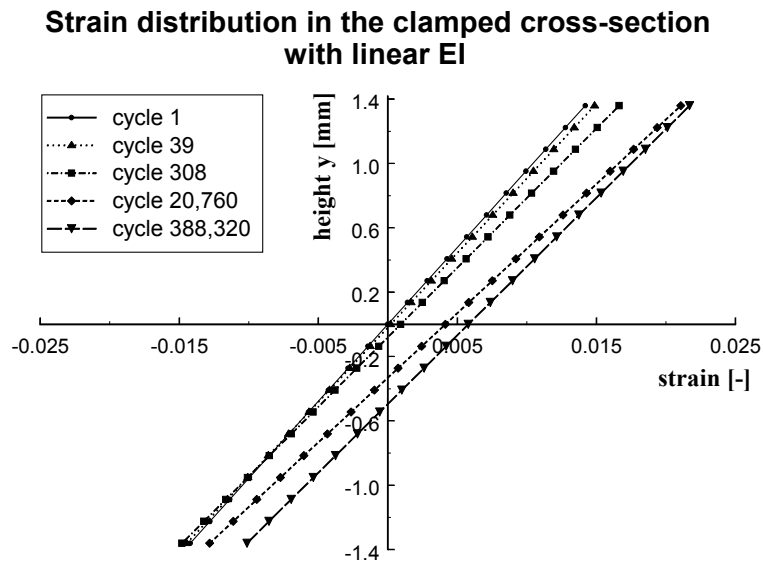


Figure 5 Simulated strain distribution in the clamped cross-section at several stages during fatigue life with linear EI.

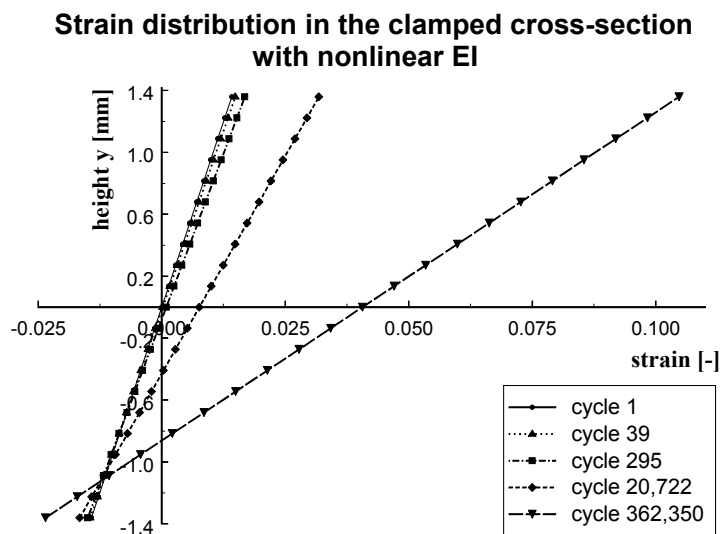


Figure 6 Simulated strain distribution in the clamped cross-section at several stages during fatigue life with nonlinear EI.

Finally, Figure 7 and Figure 8 show the corresponding simulated stress distribution in the clamped cross-section at several stages during fatigue life, with linear and nonlinear bending stiffness degradation. Again the linear simulation in Figure 7 predicts a decrease of the compressive stress with increasing damage at the tensile side and that is not realistic.

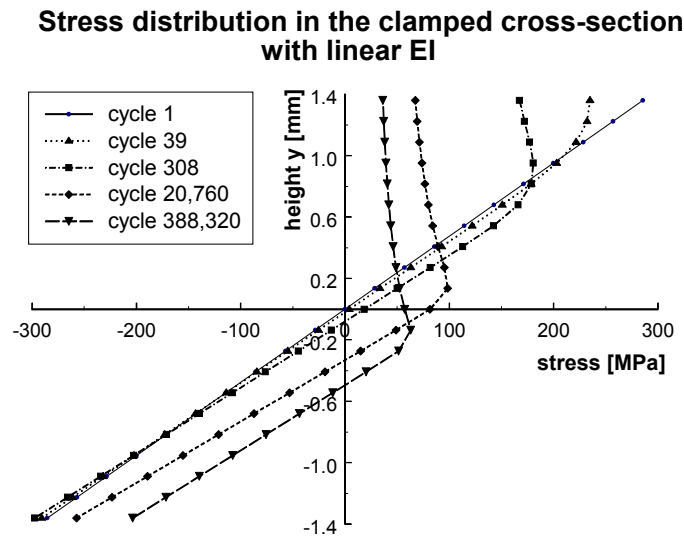


Figure 7 Simulated stress distribution in the clamped cross-section at several stages during fatigue life with linear EI.

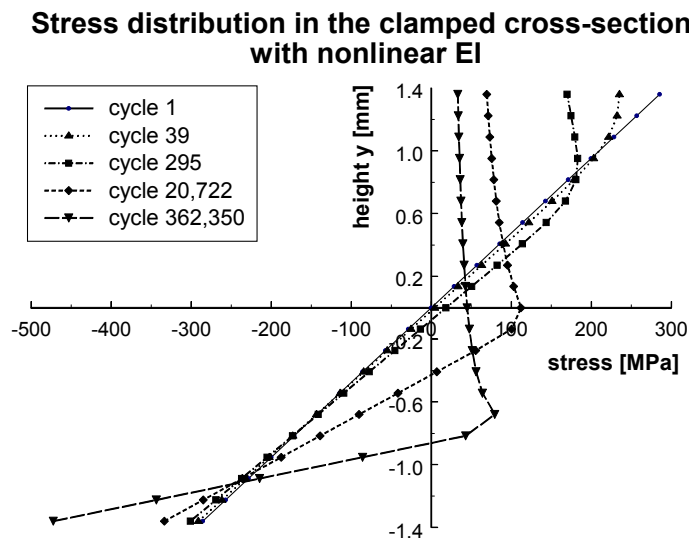


Figure 8 Simulated stress distribution in the clamped cross-section at several stages during fatigue life with nonlinear EI.

It could be argued that the final damage values are very high and would not be tolerated in fatigue design for composites. That is true, but these simulations serve the only purpose of clearly demonstrating the effect of the nonlinear correction term. Of course the larger the damage values, the larger the effect of the correction term.

6 Conclusions

The classic beam theory can be used in a modified manner to calculate the damage growth in fibre-reinforced composites during fatigue. It is reasonable to assume that if the longitudinal stiffness decreases in a linear relation with the damage, the bending stiffness will do as well. However in this paper it is demonstrated that a nonlinear correction term for the bending stiffness degradation is necessary to predict reliable results. Semi-analytical simulations of a displacement-controlled bending fatigue test are shown to support the validity of the correction.

It is evident that more sophisticated calculation tools like finite elements are widely used to simulate damage growth in composite materials, but if one wants to use a simplified tool like the modified classic beam theory, it is important to be aware of the necessity of the nonlinear correction term for the degradation of the bending stiffness.

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